Mathematical derivation

Our mathematical model is based on 1D fluid dynamic shear models (e.g. Turcotte and Schubert, 2002). We apply a constitutive equation in the form of a dislocation creep flow law

$$\dot{\gamma} = Ae^{\left(\frac{-Q}{RT}\right)}\sigma^{n}, \tag{1}$$

where $\dot{\gamma} = \partial v/\partial z$ is the shear strain rate, z is the vertical coordinate across the shear zone being zero at the base and increasing positively upwards, v is the horizontal (i.e. parallel to the shear zone) velocity, σ is the shear stress, A is a pre-exponential factor, n is the power-law stress exponent, Q is the activation energy, R is the gas constant and T is the temperature. We solve (1) for σ which yields

$$\sigma = \left[Ae^{\frac{-Q}{RT}}\right]^{-1/n}\dot{\gamma}^{1/n} \tag{2}$$

This can be more generally expressed as

$$\sigma = \mu\dot{\gamma}^{1/n} \tag{3}$$

where μ is a viscosity coefficient. We assume a normal geotherm and, hence, that the temperature increases linearly with depth inside the shear zone

$$T = T_0 - \theta z \tag{4}$$

where T_0 is the temperature at the base of the shear zone and θ is the temperature gradient. Substituting (4) into (2) provides the viscosity factor in (3):

$$\mu = A^{-1/n}e^{\frac{Q}{nR(T_0-\theta z)}} \tag{5}$$

It is difficult to derive an analytical solution for terms of the form $e^{1/z}$ as in (5). A common simplification is to use the Frank-Kamenetzky approximation which employs a term of the form e^{z}. The alternative viscosity coefficient η has then the form

$$\eta = \eta_0 e^{z/\lambda} \tag{6}$$

where η_0 is the coefficient at the base of the shear zone (z=0) and λ is the e-fold length. λ quantifies the distance over which the value of η increases by a factor of e. The material parameters in (5) derived from laboratory rock deformation experiments can be related to the new parameters η_0 and λ. The deformation is most intense at the base of the shear zone and, therefore, we want that μ is well approximated by η at the base. We require that both the values and the spatial derivatives of μ and η are identical at z=0 which provides two equations:

$$\begin{aligned} \mu_{z=0} &= \eta_{z=0} \\ \left[\frac{\partial \mu}{\partial z}\right]_{z=0} &= \left[\frac{\partial \eta}{\partial z}\right]_{z=0} \end{aligned} \tag{7}$$

The explicit form of these equations is

$$\begin{aligned} A^{-1/n} e^{\frac{Q}{nRT_0}} &= \eta_0 \\ \frac{A^{-1/n} Q\theta e^{\frac{Q}{nRT_0}}}{nRT_0^{\,2}} &= \frac{\eta_0}{\lambda} \end{aligned} \tag{8}$$

Solving the two equations yields

$$\begin{aligned} \eta_0 &= A^{-1/n} e^{\frac{Q}{nRT_0}} \\ \lambda &= \frac{nRT_0^{\,2}}{Q\theta} \end{aligned} \tag{9}$$

The expression for the stress can now be written as

$$\sigma = \eta_0 e^{\frac{z}{\lambda}} \left(\frac{\partial v}{\partial z}\right)^{1/n} \tag{10}$$

We consider a 1D horizontal shear zone where the vertical velocities are zero. We assume that horizontal pressure gradients are negligible and, therefore, the 1D force balance is (e.g. Turcotte and Schubert, 2002)

$$\frac{\partial \sigma}{\partial z} = 0 \tag{11}$$

Substituting (10) into (11) and using the two kinematic boundary conditions $v_{z=0} = v_0$ and $v_{z=W} = 0$, with W being the shear zone width, yields the velocity

$$v = \frac{v_0\left(e^{\frac{n(W-z)}{\lambda}} - 1\right)}{e^{\frac{nW}{\lambda}} - 1} \tag{12}$$

We make the solution for v dimensionless by introducing

$$\begin{aligned} \beta &= \frac{nW}{\lambda} = \frac{Q\theta W}{RT_0^{\,2}} \\ \overline{v} &= v / v_0 \\ \overline{z} &= z / W \end{aligned} \tag{13}$$

$\overline{z}$ is the non-dimensional vertical coordinate, and $\overline{v}$ is the non-dimensional horizontal velocity. The value of β depends on the thickness of the shear zone, W. Using $\theta = \Delta T / W$ with ΔT being the temperature difference between the temperature at the bottom and the top of the shear zone provides an expression for β independent on W:

$$\beta = \frac{Q}{RT_0}\frac{\Delta T}{T_0} \tag{14}$$

The value of β does not depend on the material properties n and η_0 but only on the parameters Q, ΔT and T_0. The dimensionless velocity now depends only on β:

$$\overline{v} = \frac{e^{\beta(1-\overline{z})} - 1}{e^{\beta} - 1} \tag{15}$$

The larger the value of β the stronger the strain localizes (Fig. 2).