Supplementary methodology for "A theory of glacial quarrying for landscape evolution models"

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Quarrying rate from adhesive wear theory: Equation 2 of the article

Erosion of solid surfaces in sliding contact is referred to as "wear" in materials science. Owing to asperities (bumps) on solid surfaces, the real area of contact along the sliding interface is less than the total area. The theory of adhesive wear uses the frequency of asperity-junction formation during slip and posits a probability *k* of asperity breakage at a junction to determine the volume of wear fragments Q_v:

$$Q_v = x\,k\,N\,V_e \;, \tag{S1}$$

where *x* is the slip displacement, *N* is the number of asperity junctions formed per unit distance of slip, and V_e is the volume of material eroded if an asperity breaks, which depends on the load normal to the slip surface, the hardness of the material, and the asperity geometry (Rabinowicz, 1965). In materials science applications involving microscopic asperities, *k* is a wear constant. If a specific asperity geometry and spacing are assumed, the value of *k* can be determined through experiments in which solids are slid past each other and surface erosion is measured (for example, *k* for metals is typically 0.5-160 $\times 10^{-3}$) (Rabinowicz, 1965).

For the two-dimensional stepped glacier bed of the model (Fig. 1), with steps of tread length *L*, the total number of steps is $n_s = L_T/L$, where L_T is the total length of the bed. For each unit distance of displacement, ice will traverse a step $1/L$ times, so that $N = n_s/L$, or

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$$N = \frac{L_T}{L^2}\,. \tag{S2}$$

The volume of rock eroded when a rock step fails will vary widely. Owing to ice-bed separation during sliding, cavities extend a distance S over L, so the length of the zone of ice-bed contact is $L - S$ (Fig. 1). Predominately vertical crack growth is most likely in rock where deviatoric stresses are maximized in the bed: immediately adjacent to step risers (Iverson, 1991) and at the down-glacier ends of cavities where ice reconnects with the bed (Hallet, 1996). Crack growth parallel to treads is commonly controlled by preferentially weak bedding surfaces. Thus, a reasonable approximation for rock with randomly distributed zones of weakness is that the average volume per unit width of rock eroded per step is

$$V_e = \frac{1}{2}h(L - S)\,. \tag{S3}$$

Substituting Equations S2 and S3 into S1 and dividing both sides of the resultant equation by A, yields the thickness, T, of rock eroded:

$$T = \frac{xkh}{2L}\left(1 - S/L\right). \tag{S4}$$

The time derivative of T is the rock thickness eroded by quarrying per unit time, $\dot{E}_q$, as shown in Equation 2 of the article:

$$\dot{E}_q = \frac{ukh}{2L}\left(1 - S/L\right). \tag{S5}$$

Weibull theory of rock fracture: Equation 4 of the article

Consider a brittle solid under a deviatoric stress, σ_d. The solid contains a large number of cracks. The guiding assumption of the Weibull theory of fracture is that the strength of the solid is controlled by the strength of its weakest element—presumably the element with the largest and most favorably oriented crack. Then if $P_1(\sigma_d)$ is the probability that the strength of one element is less than σ_d, the failure probability, k, of the solid can be shown to be

$$k = 1 - exp\left[-\frac{V}{V_0} P_1(\sigma_d) \right], \tag{S4}$$

where V is the volume of the solid and V_0 is a characteristic volume (Weibull, 1951; Jaeger and Cook, 1979; Bažant et al., 1991). For a fractured rock mass, V_0 is chosen to be sufficiently large to include the mass's largest cracks (Wong et al., 2006). The function $P_1(\sigma_d)$ was suggested by Weibull (1951) on the basis of experimental data:

$$P_1(\sigma_d) = \left(\frac{\sigma_d - \sigma_u}{\sigma_0} \right)^m, \tag{S5}$$

where σ_u is the smallest strength of any element, usually taken to be zero, σ_0 is the stress at which 63% of the elements fail, called the scale parameter, and m is the Weibull modulus that decreases with increasing strength heterogeneity of the solid. Thus, from (S4) and (S5),

$$k = 1 - exp\left[-\frac{V}{V_0} \left(\frac{\sigma_d}{\sigma_0} \right)^m \right]. \tag{S6}$$

This expression for failure probability is commonly used to assess rock failure (McDowell and Bolton, 1998), although some authors express σ_0^{-m}, or $V_0^{-1}\ \sigma_0^{-m}$ as a constant (Jaeger and Cook, 1979; Lu and Xie, 1995) and those focusing on rock specimens of a single size commonly omit the volume dependence (e.g., Fang and Harrison, 2002). As described in the article, Equation 4 of the article follows from Equation S6.

Figure S1A illustrates how the failure probability k depends on σ_d—normalized to σ_0 and with $V/V_0 = 1$—for values of m commonly applied to rocks (Fang and Harrison, 2002; Tang et al., 2000; Xu et al., 2004), as well as for $m = \infty$, which corresponds to an idealized rock mass of uniform strength. Values of m are obtained by fitting data to results of laboratory strength tests on many tens of specimens of a given rock type. Multiple rock specimens are subjected to testing at each of many values of σ_d, allowing the probability of failure to be determined at each stress level. Figure S1B shows data from 83 tests on biotite gneiss specimens (Lobo-Guerrero and Vallejo, 2006) and the regression that allows values of m and σ_0 to be determined.

An assumption inherent in using Equation S6 to evaluate the probability of rock-step failure is that it can be applied at scales over which it has not been tested experimentally. Ideally, multiple rock specimens of volume V_0, sufficiently large to contain the largest cracks in a rock mass, could be tested to determine m and σ_0. Such testing is not possible at scales relevant to the rock bed of Figure 1. However, if crack sizes in rocks are generally fractally distributed, as suggested by some measurements (e.g., Turcotte, 1997), then m is independent of scale because the fractal dimension of crack-size distributions implies a specific value of m (Wong et al., 2006; Lu and Xie, 1995).

A second caveat regarding Equation S6 is that it applies most strictly to a rock mass subjected to uniform, uniaxial tension (e.g., Jaeger and Cook, 1979). Although this equation can be generalized for non-uniform, triaxial stress states (Bažant et al., 1991) and the state of stress in subglacial rock steps includes such complexity (Iverson, 1991; Hildes et al., 2004), Equation S6 is used here because only a rough estimate of deviatoric stresses in the bed is available (Equation 5).

Treatment of more general distributions of step size

Size distributions of bumps or steps on glacier beds have not been systematically and comprehensively characterized, so without clear empirical direction, random, Gaussian, and fractal distributions are considered to explore the generality of the model results. For random size distributions, limiting values of step height h are chosen (0-2 m, Fig. 3, 4), with step tread length $L = 10h$. For the Gaussian size distribution, the mean value of h is 1 m, with a standard deviation of 0.25 m, such that values of h span ~0-2 m, and $L = 10h$ (Fig. 3). For the fractal distribution, the number of steps is proportional to h^{-2}, with fractal limits of 0.1 m and 2 m, and $L = 10h$ (Fig. 3). Note that steps larger than those considered herein obviously occur, but considering steps up to 5 m in height did not significantly change the results. Note also that step heights in the model must be a small fraction of the glacier thickness of a landscape evolution model for Equation 3 of the article to be valid. In all cases, ~ 5000 steps are considered and randomly positioned with respect to their size. Equations (2-5) are applied to each step i to calculate its erosion rate by quarrying $\dot{E}_{q_i}$. The rate of erosion is $\dot{E}_q = \sum_i \frac{L_i}{L_T} \dot{E}_{qi}$. Some steps are drowned by cavities. There is zero erosion of step i if $S_{i-p} \geq \sum_{j=1}^{p} L_{i-j+1}$ where p is the number steps up-glacier from step i where ice loses contact with the bed, and S_{i-p} is computed (Equation (3)) using a step height of $\sum_{j=1}^{p} h_{i-j}$.

Parameter choices

All parameters can be measured or estimated outside the model context, although some can be estimated only roughly (Table 1). The constants in the flow rule of ice are reasonably well-known: $n = 3$, which is a good assumption for ice subject to large deviatoric stresses near cavities, and $B = 73.3$ MPa $s^{1/3}$, which is a mean value based on various studies of the creep of ice at its pressure-melting temperature (Cuffey and Paterson, 2010). The largest stress that ice pressing on a ledge can support without brittle failure, σ_n^*, is considered to be 10 MPa on the basis of arguments made by Hallet (1996). Effective pressure $P_e = 0.5$ MPa, unless specified otherwise; values both well above and below this value are measured commonly beneath modern bedrock-floored glaciers (Cuffey and Paterson, 2010). The parameters m, σ_0, and V_0 are material properties of the rock bed. The range of m considered ($m = 1.5$–5.0) spans approximately the lower half of the range of values determined in most laboratory and modeling studies of rock: $m = 1$-10 (Vardar and Finnie, 1975; Tang et al., 2000; Fang and Harrison, 2002; Lobo-Guerrero & Vallejo, 2006). Only the lower half of this range is considered because engineered and seemingly more homogenous materials than most rocks, such as brick, pottery, cement, and ceramics, share the upper half of this range (McDowell et al., 1996). The value of σ_0 is taken to be 10 MPa, which is within the lower half of a range of tensile strengths reported for rocks (Jaeger and Cook, 1979) and close to some measured values (Lobo-Guerrero & Vallejo, 2006) (see also Fig. S1B). $V_0 = 10$ m^2, thereby acknowledging that cracks may be of the same scale as the steps of the simplest bed geometry considered ($h = 1$ m, $L = 10$ m). To roughly account for stresses required for subcritical crack growth, $\kappa = 1/3$, equal to the ratio of stress-corrosion limit to fracture toughness used in Hallet's (1996) quarrying model.

Specifying c (Equation 5) with confidence is particularly difficult due to non-uniform normal stresses over zones of ice-rock contact and resultant non-uniformity of deviatoric stresses in steps (Iverson, 1991). This problem is compounded by the likely departure from isotropic elasticity in rock steps with large pre-glacial cracks. Nevertheless, results of contact problems, in which a uniform or non-uniform normal stress is applied over a finite footprint to the surface of a semi-infinite elastic half-space (Lawn and Wilshaw, 1975) or quarter space (Hetényi, 1960) provide some guidance. They indicate that over most of the zone of contact, c will be a small fraction of the normal stress on the contact, except very locally—for example, immediately adjacent to its up-glacier edge where bed-parallel tensile stresses at the bed surface indicate $c = 0.68$ for the case of uniform loading (Hetényi, 1960; Hallet, 1996). Here I use a smaller value more consistent with generally compressive principal stresses beneath the zone of contact (Iverson, 1991; Hildes, 2001): $c = 0.1$.

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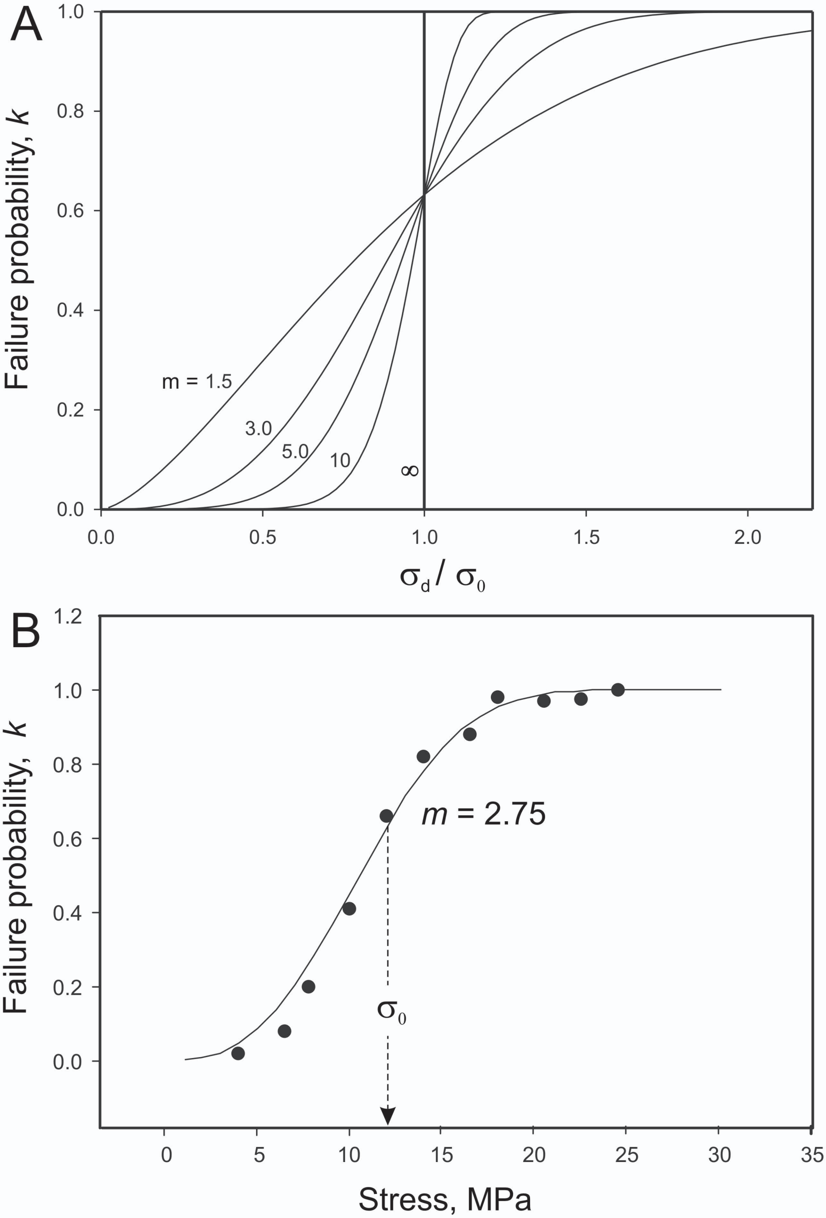

Figure S1. Weibull failure probability distributions. (A) Distributions for various values of m, with σ_d normalized to σ_0 and with $V/V_0 = 1$ (modified from Jaeger and Cook, 1979). (B) Regressed data from tests on 83 biotite gneiss specimens, indicating $m = 2.75$ and $\sigma_0 = 12$ MPa (plotted from data of Lobo-Guerrero and Vallejo, 2006).