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APPENDIX: Obtaining Soil Flux from Integration of the Soil Production Rate

PRINCIPLES AND PRACTICE

The magnitude of the soil flux $h\bar{\mathbf{q}}$ [L^2 t^{-1}] can be estimated from downslope integration of the local soil production rate p_η [L t^{-1}] between two arbitrarily curved "flow lines" of downslope soil motion. Here we first describe this integration as applied to sites at Nunnock River, Point Reyes and Tennessee Valley. We then examine possible sources of error in our estimates of soil flux arising from the numerical integration and from the assumption that the rate of change in soil storage is negligible.

To illustrate the idea behind the downslope integration, first consider two soil transport flow lines that are parallel and straight and separated by a uniform contour distance B. Let x denote a horizontal axis that is positive in the downslope direction. The z-axis is vertical and positive upward. Neglecting horizontal tectonic motion the vertically integrated equation of mass conservation is:

$$\frac{\partial}{\partial x}\left(Bh\bar{q}_x\right) + \frac{\partial}{\partial t}\left(Bh\bar{c}\right) + Bc_\eta p_\eta = 0\,, \tag{A.1}$$

where $\bar{q}_x$ is the vertically averaged soil flux density parallel to x, h is the soil thickness, $\bar{c}$ is the vertically averaged soil concentration, c_η is the soil concentration at the base of the soil, p_η is the rate of soil production, and t is time. Note also that, although B could be removed from (A.1), we retain it here for illustration. Integrating this with respect to x from the divide ($x = 0$) to a position $x = X$,

$$Q(X,t) = -B\int_0^X \frac{\partial}{\partial t}(h\bar{c})\,\mathrm{d}x - B\int_0^X c_\eta p_\eta\,\mathrm{d}x\,, \tag{A.2}$$

which illustrates that the total soil flux $Q(X)$ [L^3 t^{-1}] at position $x = X$, that is $Q(X) = Bh\bar{q}_x|_X$, obtains by integrating the unsteady term and the soil production rate upslope of X. In the case of steady soil thickness h, and uniform and constant concentrations $\bar{c}$ and c_η, this reduces to

$$Q(X) = -Bc_\eta\int_0^X p_\eta\,\mathrm{d}x\,. \tag{A.3}$$

If the production rate p_η is uniform, then the soil flux $Q(X)$ increases linearly with X. Note that the first integral quantity in (A.2) is equivalent to integrating the local rate of change in soil storage, $\partial(h\bar{c})/\partial t$, over the area $A(X) = BX$. Similarly, the second integral quantity in (A.2) and the integral quantity in (A.3) are equivalent to integrating the local soil production rate over the area $A(X) = BX$. Also note that the soil flux per unit contour distance at position X is $Q(X)/B$. The formulation above can be generalized to a curvilinear or radial coordinate system (Furbish, unpublished notes), although it suffices here to proceed to a simpler formulation that builds on these points.

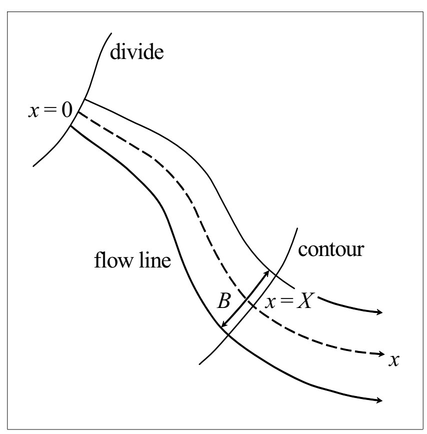

Figure A1. Curvilinear coordinate system; flow lines normal to contours.

Consider two soil transport flow lines that are everywhere normal to elevation contours (Fig. A1). For convenience we now let x denote a curvilinear downslope coordinate centered between the two flow lines with origin ($x = 0$) at the upslope divide. At any coordinate position x there is an arbitrarily curved elevation contour segment, locally normal to x, with length B between the two flow lines. Thus, $B = B(x)$.

The total soil flux Q [$L^3\ t^{-1}$] passing a given contour segment at position $x = X$ must equal the total rate of soil production, minus the rate of change in storage, upslope of the segment between the flow lines. Thus,

$$Q(X,t) \;=\; -\int_{A(X)} c_\eta p_\eta \,\mathrm{d}A \;-\; \int_{A(X)} \frac{\partial}{\partial t}(h\bar{c})\,\mathrm{d}A\,, \tag{A.4}$$

indicating that $Q(X, t)$ is obtained by simply integrating the local production rate and the rate of change in soil storage over the area $A(X)$ upslope of $x = X$.

As described in the next section, we assume that the soil storage term in (A.4) is negligible relative to the other terms whence,

$$Q(X) \;\approx\; -c_\eta \int_{A(X)} p_\eta \,\mathrm{d}A\,, \tag{A.5}$$

where c_η is assumed to be uniform. With square brackets denoting an average, the magnitude of the soil flux per unit contour distance at position $x = X$ is then estimated as an average over $B(X)$, namely $[h\bar{q}_x]|_X = Q(X)/B(X)$, which is the value that we report in the text.

The numerical integration of (A.5) is performed as follows. Voronoi polygons are constructed for soil thickness measurements within and near the total area $A(X)$. Let a_i denote the subarea of the ith polygon falling within $A(X)$ such that $A(X) = \sum a_i$. Then assuming the local soil production rate $p_\eta = -P\exp(-h/\gamma)$,

$$Q(X) \;\approx\; c_\eta P \sum_i^N a_i \exp(-h_i/\gamma)\,, \tag{A.6}$$

where N is the total number of polygon subareas. The importance of constructing Voronoi polygons is that this procedure objectively weights each local production rate (obtained from measured soil thickness) in proportion to its relative areal (Voronoi) coverage within $A(X)$.

We also note that reported slopes are averages obtained at position X. That is, we estimated local slopes for two to seven locations along B (depending on its length), then averaged these to obtain $S(X)$. The significance of this is that plots involving flux, slope and the depth-slope product are based on averages over $B(X)$ rather than representing "local" values.

TRANSIENT SOIL STORAGE

Nonuniform soil thicknesses on the hillslopes suggest that transient soil storage is non-zero. Nonetheless we suggest that the storage term in (A.4) is significantly smaller than the other terms, with the implication that (A.5) is an adequate estimate of the soil flux.

Under steady-state conditions, where either uplift is balanced by stream incision at the lower hillslope boundary or the land-surface is lowering uniformly, the storage term in (A.4) is zero (for constant $\bar{c}$). Changes in storage are thus related either to changes in the lower boundary condition wherein effects of this condition are propagated upslope, or to changes in the soil transport rate due to a change in the transport coefficient (or diffusivity), or both. The sites were selected to avoid complications related to possible changes in transport processes, so here we focus on the magnitude of soil storage related to changes in the lower boundary condition. The downslope thickening soils at the field sites in particular are consistent with cessation of stream downcutting, whence soils thicken with time, and this thickening slowly propagates upslope.

Assuming momentarily that the soil flux is proportional to slope, namely $Q(X) = -BDS$ where D [L^2 t^{-1}]is a diffusivity, then with uniform c_η and constant $\bar{c}$, (A.4) integrates to

$$BDS(X,t) = c_\eta A\langle p_\eta\rangle + \bar{c}A\frac{\partial\langle h\rangle}{\partial t}, \tag{A.7}$$

where chevron brackets denote that the quantity is averaged over the area A. Under steady conditions this reduces to

$$BDS(X) = c_\eta A\langle p_\eta\rangle . \tag{A.8}$$

The terms in (A.7) can be directly scaled to evaluate their relative magnitudes. To clarify the physical basis for this scaling, however, we consider the rate of change of these terms in response to a change in the lower boundary condition.

Envision a change in the transport rate BDS during a small interval dt, following a steady state condition at time t. Expanding (A.7) as a Taylor series about t,

$$\begin{aligned} BDS(X,t+\mathrm{d}t) &= BDS(X,t) + BD\frac{\partial S}{\partial t}\mathrm{d}t \\ &= c_\eta A\langle p_\eta\rangle(t) + c_\eta A\frac{\partial\langle p_\eta\rangle}{\partial t}\mathrm{d}t + \bar{c}A\frac{\partial^2\langle h\rangle}{\partial t^2}\mathrm{d}t + \mathrm{O}[(\mathrm{d}t)^2] . \end{aligned} \tag{A.9}$$

Thus, the new transport rate at time t + dt involves the "old" steady-state balance at time t; and the change in the transport rate is balanced by changes in production or storage, or both. Thus, to clarify when the storage term can be neglected, it suffices to show when the term involving the second derivative is small relative to terms involving first derivatives. More simply, the rate of change of (A.7) is

$$BD\frac{\partial S}{\partial t} = c_\eta A\frac{\partial\langle p_\eta\rangle}{\partial t} + \bar{c}A\frac{\partial^2\langle h\rangle}{\partial t^2}, \tag{A.10}$$

which may be viewed as a measure of the extent to which a change in soil production is accommodated by a change in transport versus being partitioned into storage.

Let $S = \partial\zeta/\partial x$, where ζ is the land-surface elevation. Then noting that $\partial\langle p_\eta\rangle/\partial t = \partial\langle -P\exp(-h/\gamma)\rangle/\partial t \approx (1/\tau)\partial\langle h\rangle/\partial t$, where $\tau = \gamma/\langle p_\eta\rangle$ is a measure of the mean soil residence time,

$$BD\frac{\partial}{\partial t}\left(\frac{\partial\zeta}{\partial x}\right) = \frac{c_\eta A}{\tau}\frac{\partial\langle h\rangle}{\partial t} + \bar{c}A\frac{\partial^2\langle h\rangle}{\partial t^2} . \tag{A.11}$$

The slope and the soil thickness must change over the same timescale T in response to a change in the lower boundary condition. We thus define the following dimensionless quantities denoted by a circumflex:

$$\zeta = \gamma\hat{\zeta}, \quad \langle h\rangle = \gamma\hat{h}, \quad x = \lambda\hat{x} \text{ and } t = T\hat{t} . \tag{A.12}$$

Substituting these first into (A.8) obtains

$$\frac{\partial\hat{\zeta}}{\partial\hat{x}}\Big|_{x=X} = c_\eta \frac{T^*}{\tau}\,, \tag{A.13}$$

where $T^* = \lambda A/BD$ is like a relaxation timescale. (Note that, in the case of parallel, straight flow lines, $A = BX$, whence $T^* = \lambda X/D$.) Moreover, this steady-state case requires that $T^* \sim \tau$.

Turning to (A.11),

$$\frac{\tau}{T^*}\frac{\partial}{\partial x}\left(\frac{\partial\hat{\zeta}}{\partial\hat{t}}\right)\Big|_{x=X} = c_\eta \frac{\partial\hat{h}}{\partial\hat{t}} + \bar{c}\frac{\tau}{T}\frac{\partial^2\hat{h}}{\partial\hat{t}^2}\,, \tag{A.14}$$

which indicates that the last term can be neglected if $\tau \sim T^* \ll T$. Qualitatively, inasmuch as transport depends on land-surface slope, any effect on transport due to a change in the lower boundary condition is felt through changes in slope that propagate upslope. For a relaxing hillslope, a change in transport rate (i.e. a change in slope) generally involves a change in soil thickness (and a concomitant change in soil production), which implies a change in storage. Nonetheless, inasmuch as changes in soil thickness occur over a timescale that is much longer than the mean soil residence time, soil production remains essentially balanced by transport.

For completeness we substitute (A.12) into (A.7) with $\tau \sim T^* = \lambda A/BD$ to obtain:

$$\frac{\partial\hat{\zeta}}{\partial\hat{x}}\Big|_{x=X} - c_\eta = \bar{c}\frac{\tau}{T}\frac{\partial\hat{h}}{\partial\hat{t}} = \bar{c}\frac{\lambda A}{BDT}\frac{\partial\hat{h}}{\partial\hat{t}}\,. \tag{A.15}$$

Based on the scaling quantities adopted in (A.12) and applied in (A.15), the timescale T may be interpreted as the time that it takes to accumulate a soil thickness equal to $\gamma \sim h$. In the absence of transport — assuming that all production goes into storage — then $T \sim \tau$, typically on the order of several thousand years, the shortest possible value of T. Numerical simulations of relaxing hillslopes suggest, however, that to accumulate (excess) soil thicknesses equal to $\gamma \sim h$ requires periods approaching the relaxation time of the hillslope, $T_R \sim L^2/D$, where L is the hillslope length (Furbish, 2003; Furbish and Dietrich, in prep.).

The scaling in (A.15) also reveals important consequences of land-surface slope and gradient. The lengthscale λ may be interpreted as the distance over which the land-surface elevation changes by an amount $\Delta\zeta \sim \gamma$. Thus, as λ increases (slope decreases), the timescale T^* increases such that the magnitude of the storage term increases. This merely reflects that, with decreasing slope and therefore decreasing soil throughput, the mean residence time increases. In addition, for given area A, a small ratio A/B coincides with diverging flow lines, whereas a large ratio A/B coincides with converging flow lines. Thus, the significance of the storage term decreases with increasing divergence of the lines, whereas this term may become increasingly important with converging flow lines.

A similar scaling can readily be applied with the assumption that the soil flux is proportional to the product of soil thickness and land-surface slope, namely $Q(X) = -BK\langle h\rangle S$ where K [L t^{-1}] is a transport coefficient. Using (A12) with $\tau \sim T^* = \lambda A/BK\gamma$ we obtain:

$$\hat{h}\frac{\partial\hat{\zeta}}{\partial\hat{x}}\Big|_{x=X} - c_\eta = \bar{c}\frac{\tau}{T}\frac{\partial\hat{h}}{\partial\hat{t}} = \bar{c}\frac{\lambda A}{BK\gamma T}\frac{\partial\hat{h}}{\partial\hat{t}}\,. \tag{A.16}$$

Comparing this with (A.15) it is apparent that conclusions regarding the importance of the storage term are the same as those outlined above.

SPURIOUS CORRELATION BETWEEN FLUX, SLOPE AND THICKNESS

If the hypothesis that soil flux is linearly proportional to land-surface slope is correct, then a plot of flux $[h\bar{q}_x]|_X = Q(X)/B(X)$ versus slope $S(X)$ should in principle exhibit a linear trend with zero intercept and slope equal to the diffusivity D. Similarly, if the hypothesis that soil flux is proportional to the product of soil thickness and land-surface slope is correct, then a plot of flux $[h\bar{q}_x]|_X$ versus the product $h(X)S(X)$ should exhibit a linear trend with zero intercept and slope equal to the transport coefficient K (Fig. 3A). However, as an integrated quantity, th soil flux, see (A.5), must generally increase with downslope distance. Moreover, both slope and thickness generally increase downslope at the field sites. This means that plots of flux versus slope, or flux versus the product of thickness and slope, may exhibit spurious (positive) correlations, and therefore do not represent a rigorous "test" of the two hypotheses. For this reason we consider plots involving the ratios $[h\bar{q}_x]|_X/S(X)$ and $[h\bar{q}_x]|_X/h(X)S(X)$ versus distance X (Figure 3B). The effect of this is to remove correlations among flux, slope and thickness. For the data to be consistent with one of the transport hypotheses, the trends in these plots should be flat, at a value equal to either D or K. See Data Repository Figures A2 and A3, below, for plots of flux versus gradient and flux/gradient.

REFERENCES CITED

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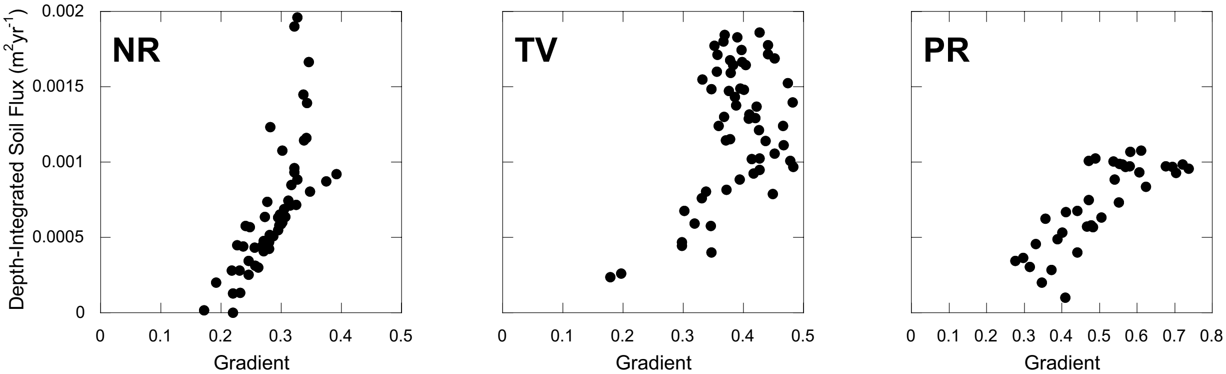
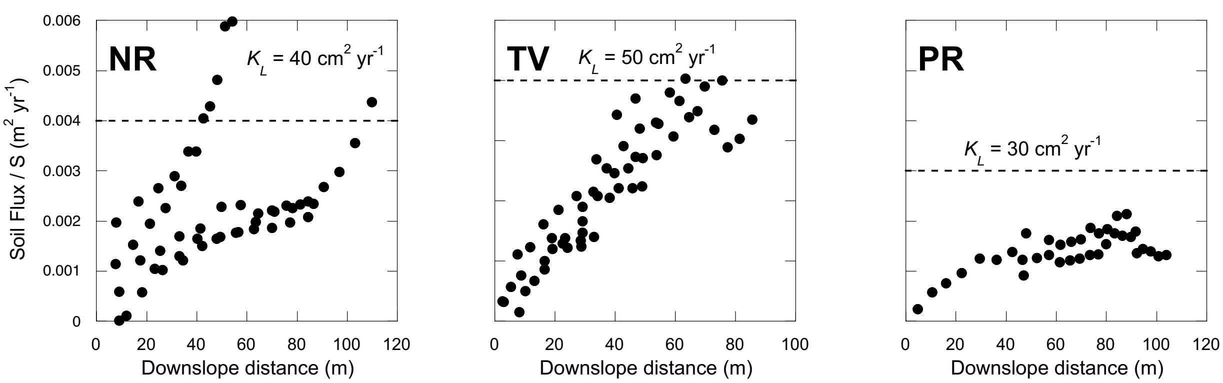


Data Repository Figure A2: Depth-integrated soil flux used in Figure 3 in text (calculated as described above) per unit contour length (m^2yr^{-1}) versus the gradient for all field sites.
For data to fit the linear slope-dependent transport law both a linear increase of flux with gradient, as well as an intercept with the origin (i.e. zero flux at zero slope) are required.
NR-Nunnock River; TV-Tennessee Valley; PR-Point Reyes.



Data Repository Figure A3: Depth-integrated flux divided by gradient versus downslope distance. K_L value is dashed line independently determined by several different studies: NR value from Heimsath et al. (2000); TV value from Reneau (1988), used in Dietrich et al. (1995) and Heimsath et al. (1997, 1999); PR value from Reneau (1988).
Used in a similar way to Figure 3B in the text, these data would support a linear slope-dependent transport law if the data were homoscedastic about the transport coefficient plotted as the dashed line for each site.
NR-Nunnock River; TV-Tennessee Valley; PR-Point Reyes.