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Analytic Solutions to the Diffusion Equation for Cinder Cone Evolution

The diffusion equation in polar coordinates with radial symmetry is

$$\frac{\partial h}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(\kappa r \frac{\partial h}{\partial r}\right) \tag{1.1}$$

If κ is a constant, Equation (1.1) becomes

$$\frac{\partial h}{\partial t} = \kappa\left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r}\right) \tag{1.2}$$

The solution to Equation (1.2) with an initial radially-symmetric topography $f(r)$ and a constant-elevation boundary condition $h(a,t) = 0$ at $r = a$ is given by

$$h(r,t) = \frac{2}{a^2}\sum_{n=1}^{\infty} e^{-\kappa\alpha_n^2 t}\frac{J_0(\alpha_n r)}{J_1^2(\alpha_n a)}\int_0^a r f(r) J_0(\alpha_n r)dr \tag{1.3}$$

where $\alpha_n, n = 1, 2, ...$ are the positive roots of $J_0(\alpha a) = 0$ and $J_0(r)$ and $J_1(r)$ are Bessel functions of the first and second kind (Culling, 1963). This solution utilizes two boundary conditions. First, at $r = 0$ Equation (1.3) implicitly assumes

$$\left|\frac{\partial h}{\partial r}\right|_{r=0} = 0 \tag{1.4}$$

because Equation (1.3) is a series comprised of even functions of radius (i.e. $J_0(r)$). The Bessel function $J_0(r)$ has a derivative of zero at $r = 0$, so a series comprised of a sum of these functions must also obey that condition. The second boundary condition is a constant elevation of zero at $r = a$. For simplicity we assumed the elevation of the surrounding alluvial flat equal to zero, but any constant value could be used. In hillslope geomorphology, a fixed-elevation boundary condition applies to a channel that is capable of transporting all of the sediment delivered by the hillslope, resulting in neither deposition nor erosion at the channel location. In the cinder cone case, a fixed elevation at $r = a$ could correspond to a channel that wraps around the base of the cone, but this is not a common occurrence. In most cases, volcanic cones are surrounded by alluvial flats that do not readily transport material from the base of the cone. Equation (1.3) can still be used for these cases, but a must be chosen to be much larger than the radius of the cone. This way, debris will be removed from the cone and deposited on the surrounding flat. In these cases, the boundary condition at $r = a$ will simply serve to maintain the alluvial flat or piedmont at a constant elevation very far from the base of the cone.

For a volcanic cone of radius r_c, crater-rim radius r_r, colluvial fill radius of r_f, and maximum height of h_0 (Figure 1.1), $f(r)$ is given by

$$f(r) = \left\{\begin{array}{rll} h_0\left(1 + \frac{r_f - r_r}{r_c - r_r}\right) & \text{if} & r < r_f \\ h_0\left(1 + \frac{r - r_r}{r_c - r_r}\right) & \text{if} & r_f \geq r < r_r \\ h_0\left(1 + \frac{r_r - r}{r_c - r_r}\right) & \text{if} & r_r \geq r < r_c \\ 0 & \text{if} & r \geq r_c \end{array}\right\} \tag{1.5}$$

As a technical point, it should be noted that the colluvial fill radius r_f must be finite in order for Equation (1.3) to apply because otherwise the boundary condition given by Equation (1.4) is violated. Physically,

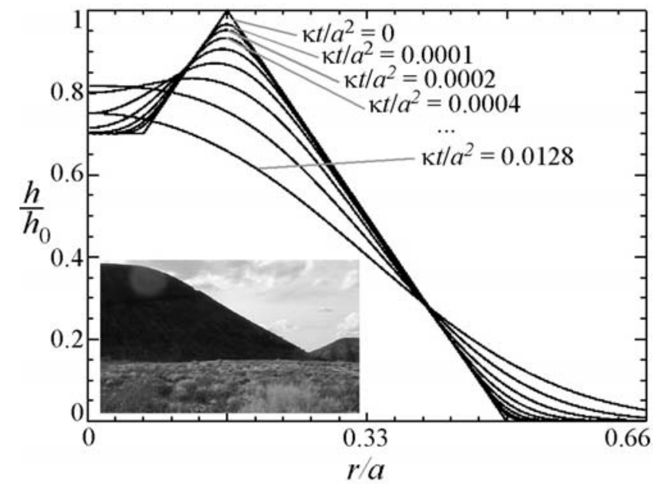

Fig. 1.1. Plots of analytic solutions to the diffusion equation for a volcanic cinder cone for a range of times following eruption.

r_f corresponds to the width of the colluvium that fills the crater shortly following eruption. Substituting Equation (1.5) into Equation (1.3) gives

$$h(r,t) = \frac{2h_0}{a^2}\sum_{n=1}^{\infty} e^{-\kappa\alpha_n^2 t}\frac{J_0(\alpha_n r)}{\alpha_n J_1^2(\alpha_n a)}\left[\left(\frac{r_f^2}{r_c-r_r}J_1(\alpha_n r_f) - \frac{2r_r}{r_c-r_r}J_1(\alpha_n r_r) + \left(1+\frac{r_r}{r_c-r_r}\right)r_c J_1(\alpha_n r_c)\right)\right.$$
$$\left. + \frac{\alpha_n}{r_c-r_r}\left(2\int_0^{r_r} r^2 J_0(\alpha_n r_r)dr + \int_0^{r_f} r^2 J_0(\alpha_n r_f)dr + \int_0^{r_c} r^2 J_0(\alpha_n r_c)dr\right)\right] \quad (1.6)$$

The three integrals that appear in Equation (1.6) can also be written as infinite series. In practice, however, it is more accurate to evaluate those integrals numerically because the series converge very slowly. Therefore, although closed-form analytic solutions for volcanic cones can be written down, calculating and plotting the solutions requires some numerical work.

First, we must solve for the roots of $J_0(\alpha a) = 0$. This is done numerically using a root-finding technique. The Bessel function has an infinite number of roots, so how many do we need to compute? For our purposes, the first one hundred roots are adequate, but more roots may be required for high-resolution profiles or very young cones. Second, we must evaluate the integrals in Equation (1.6) numerically. The integrals in Equation (1.6) were computed using the qsimp routine of Press et al. (1992).

Figure 1.1 illustrates radial profiles of Equation (1.6) for the initial cone and at eight subsequent times from $\kappa t/a^2 = 0.0001$ to $\kappa t/a^2 = 0.0128$. Both axes are normalized. The y-axis is normalized to the initial cone height h_0. As in all diffusion problems, the solution can be scaled up or down in height with no change in the relative cone shape. The r-axis is scaled to the model domain length a.

In the cones early-stage evolution, the greatest change occurs at the crater rim. This is not surprising since this is where the profile curvature is greatest. For intermediate times, the position of the crater rim migrates inward. This migration is associated with the additional advective term in Equation (1.1) that is not present in 2D diffusion problems. Eventually, the crater is filled with debris and the late-stage cone morphology is described by $J_0(ar)$ with decreasing amplitude over time. At this point, the exponential time-dependent term in Equation (1.3) has filtered all of the high-frequency components in the topography and only the lowest-frequency term is significant in the series. The inset photo in Figure 1.1 shows the profile of an early Quaternary volcanic cone in the Cima volcanic field, California. The profile shape is similar to the late-stage profiles plotted in Figure 1.1.

Figure 1.2 compares the observed topography of two cones in the Crater Flat volcanic field, Nye County, Nevada, to best-fit model solutions. Lathrop Wells Cone has been radiometrically dated to be 77 ka, while Black Cone has an age of 1.1 Ma. A nonlinear diffusion model is needed to properly resolve the slope rotation component of cone evolution, but the linear model can be fit to observed data by assuming an initial angle comparable to angles observed on young cones such as Lathrop Wells. Model solutions compare well to the observed profiles, and best-fit morphologic ages are 400 and 4000 m^2, respectively. The ratio of the

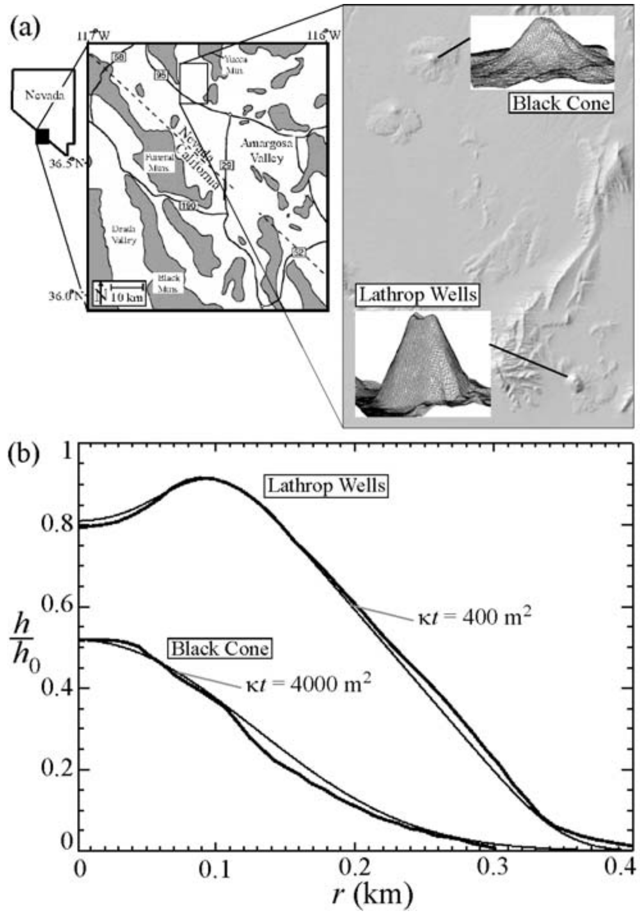

Fig. 1.2. (b) Observed (thick line) and best-fit model profiles (thin line) for a relatively young (Lathrop Wells Cone) and a relatively old (Black Cone) cone in the Crater Flat volcanic field, Nevada (location map in (a)).

morphologic age to the radiometric age provide estimates for the time-averaged κ values for these cones: 5.2 and 3.6 m^2/kyr. These values are at the upper range of κ values inferred from pluvial shoreline and fault scarps in the southwestern U.S.