Supplementary Material for 'Emergent simplicity despite local complexity in eroding fluvial landscapes'

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Summary

This Supplementary Information contains two movies (details follow) and a simple mathematical explanation for how models of physical erosion can be simplified to very few parameters. The simple (few parameter) model is amenable to a straightforward, computationally inexpensive, exploration of parameter space at much larger scales. For example, Figure 2 shows the results of running the model where the evolution of 10^5 blocks is predicted for 10^5 time steps, which takes 50 s using a 2.6GHz Intel Core i7 processor. Finally, results showing the effect of changing the critical threshold value, c, are given in Figure 3 of this document. Results are described in the main manuscript and the movies help to show the time dependent behaviour.

Movies

Movie 1 shows the time dependent evolution of solutions to Equation (3) in the main manuscript for constant critical toppling height, c. The upper panel and inset show the evolution of the river coloured by timestep. The inset panel shows the region contained within the black box shown in the main panel. The rectangular panels below show relief along the river as a function of time, Δz, and relief greater than the critical value for toppling. The square panels below show frequency (black bars) and cumulative frequency (red curves) of relief. Solutions for the same model are also shown in Figure 1d-i of the main manuscript and as red solid and dotted curves in Figure 3b of this document. Movie 2 shows the distribution of relief generated by running this model 100 times with random (but uniformly distributed) starting conditions.

Simplifying a physical model of block toppling

The following describes how physical models of erosion along rivers can be described as a consequence of thresholds. The resultant simple models have very few parameters. In the main manuscript a simple (few parameter) model is explored for insights into the evolution of fluvial landscapes from very small (meter) to large (tens to hundreds of kilometres) scales.

Physical erosion is a consequence of body or surface forces (F) being sufficiently large that erosional thresholds, c, are exceeded. More formally, in discrete notation, at any position along a river, x, elevation will change as function of time, t, such that

$$z_{t+1}^x = \begin{cases} z_t^x & \text{if} \quad F \le c \\ z_t^x - \Delta z & \text{if} \quad F > c, \end{cases} \qquad (1)$$

where Δz is change in elevation, which can be set by, for example, the size of the rock mass (e.g. pebble, basalt column, fractured schist) being moved between time t and $t+1$. This simple description could be expanded to incorporate, for example, shear stresses or drag and critical thresholds for sliding, saltation, toppling or fracturing. The simple model appears to be a universal description of physical erosion along rivers. This supplementary document shows

one way in which a simple physical model of blocks toppling (e.g. Lamb & Dietrich, 2008; Stucky de Quay et al., 2019), which appears to be a reasonably description of fluvial erosion in regions of exposed bed rock, can be reduced to a simple model in which erosion occurs if rock column height exceeds a critical value for toppling (i.e. $\Delta z > c$). It is straightforward (and computationally efficient) to expand this model so that the consequences of local physical erosion for fluvial erosion at much larger scales can be explored. Simplification of other well known erosional models (wear; transport-limited erosion) are also examined.

In the simple scheme explored here, the propensity of columns of rock to topple is estimated as a function of drag, shear stress, rock mass and buoyancy. The force generated by drag on the (unit width) column of rock can be expressed as

$$F_d = \frac{1}{2}\rho_w C_d u^2 h_1, \tag{2}$$

where ρ_w is density of water, C_d is the dimensionless drag coefficient, u is water velocity, h_1 is height of the column exposed to flowing water. For reasonable values of parameters (see Table 1) in Equation (1), F_d is O($10^3 - 10^6$) N for a column of unit width. The force generated by shear at the top of the unit width column can be expressed as

$$F_\tau \approx \rho_w g h_2 \frac{\mathrm{d}z}{\mathrm{d}x} L, \tag{3}$$

where g is gravitational acceleration, h_2 is depth of the flowing water, $\mathrm{d}z/\mathrm{d}x$ is channel bed slope, and L is width of the column. F_τ is expected to be O($10 - 10^3$) N for slopes between O($10^{-3} - 10^{-2}$). The buoyancy force generated as a result of water displaced by the column of unit width rock can be expressed as

$$F_b = \rho_w g L h_3, \tag{4}$$

where h_3 is depth of the water at the base of the column. F_b is expected to be up to O(10^5) N. The force exerted by the column of unit width rock is

$$F_g = \rho_r g L H, \tag{5}$$

where ρ_r is density of the rock column. F_g is expected to be up to O(10^5) N.

Calculating moments (see Figure 1) generated by application of these forces indicates that the column of rock will topple if

$$2HF_\tau + F_d\left(2H - h_1\right) + LF_b \geq LF_g. \tag{6}$$

Substituting Equation (4) into (6) and rearranging to make column height the subject yields

$$H\left[2F_\tau + 2F_d - L^2\rho_r g\right] \geq h_1 F_d - LF_b. \tag{7}$$

If $2F_\tau + 2F_d \geq L^2\rho_r g$, the column will topple if,

$$H \geq \frac{h_1 F_d - LF_b}{2F_\tau + 2F_d - L^2\rho_r g}. \tag{8}$$

If $2F_\tau + 2F_d < L^2\rho_r g$, the column will topple if,

$$H \leq \frac{h_1 F_d - LF_b}{2F_\tau + 2F_d - L^2\rho_r g}. \tag{9}$$

The right hand side of Equation (9) is less than unity for the parameter values given in Table 1. In other words blocks are likely to be stable if $2F_\tau + 2F_d < L^2 \rho_r g$. We therefore focus on Equation (8). It is desirable to recast this equation in terms of elevation, z. For simplicity, if we assume that the right hand side of Equation (8) is constant, c, the evolution of longitudinal river profile elevations can then be expressed as

$$z_{t+1}^x = \begin{cases} z_t^x & \text{if} \quad \Delta z \leq c \\ z_t^x - \Delta z & \text{if} \quad \Delta z > c, \end{cases} \tag{10}$$

where $H = \Delta z$ (i.e. change in relief between adjacent columns; $\Delta z = z_t^x - z_t^{x-1}$), and x is position along the river. Solutions to Equation (10) are shown in the main manuscript and below for different starting conditions and distributions of c.

Examples of simplifying alternative erosional models

There are many ways in which river beds lower including by removal of alluvium or abrasion of bedrock. It seems likely that many erosional processes can be recast in a similar form to Equation (10). For example, if we consider erosion by wear, following Lamb et al. (2008)'s recasting of Cutter's (1960) classic impact wear model, the volume of bedrock eroded due to wear can be expressed as $V_i = V_p \rho_s w^2 / 2\epsilon$. V_p, ρ_s and w are the respective volume, density and impact velocity of particles (normal to the bed; e.g. saltating sediment). ϵ is the 'deformation wear factor', in other words the amount of energy required to remove a unit volume of eroded rock by wear, which incorporates the capacity of bedrock to store energy elastically. Note that, following Lamb et al. (2008), in this example there is no threshold kinetic energy for erosion to occur, except that the kinetic energy ($V_p \rho_s w^2 / 2$) must be greater than zero. For this simple scheme Equation (10) can be rewritten as

$$z_{t+1}^x = \begin{cases} z_t^x & \text{if} \quad V_p \rho_r w^2 / 2 \leq c \\ z_t^x - \Delta z & \text{if} \quad V_p \rho_r w^2 / 2 > c, \end{cases} \tag{11}$$

where c is 0 and Δz is V_i/A; A is the area of eroded bed rock removed. Clearly some of the scalings in this model are different to those considered in the block toppling model, however, the overarching rule (i.e. lowering occurs once a threshold has been exceeded) remains the same.

Perhaps more speculatively, if we consider transport-limited erosion, e.g. lowering of river profiles by movement of alluvium currently at rest, we can recast Equation (10) as

$$z_{t+1}^x = \begin{cases} z_t^x & \text{if} \quad \tau < c \\ z_t^x - \Delta z & \text{if} \quad \tau \geq c, \end{cases} \tag{12}$$

where $c = (\rho_r - \rho_w)gD$, i.e. we assume movement initiates at the Shields number, $\tau_* = \tau/c$. An important complexity is that Δz is likely to scale with shear stress and at short timescales it is expected to be a fraction of the diameter of the characteristic particle being moved, D (e.g. Wong & Parker, 2006).

All of these schemes can be made more complex (complete), for example, we might combine them, consider angular impingement of water or rock particles on bed rock, cohesive strength of joints, disentrainment of sediment, etc. It seems likely that in many models of physical erosion there is a critical threshold to overcome for erosion to initiate, which indicates that Equation (1) is perhaps a reasonable general representation of fluvial erosion.
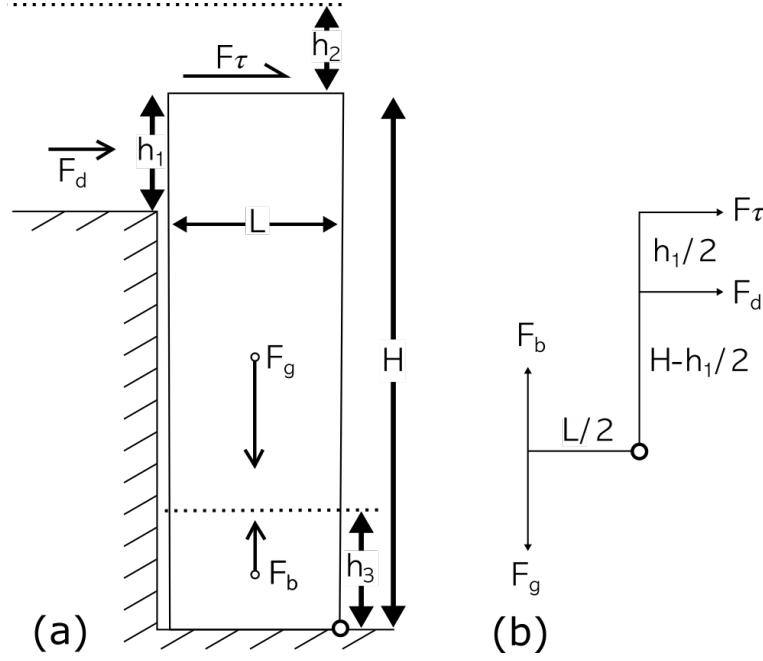
Figure 1: **Schematic block toppling.** (a) H and L = height and length of rock column. F_d = drag force on column exerted over length h_1. F_τ = shear force; h_2 = depth of water flowing across top of column. F_g = body force exerted by rock column. F_b = buoyancy force; h_3 = depth of displaced water at base of column. ∘ = pivot for moments calculations. (b) Schematic for torque calculation.

Table 1: Parameters and their values used for moments calculations.

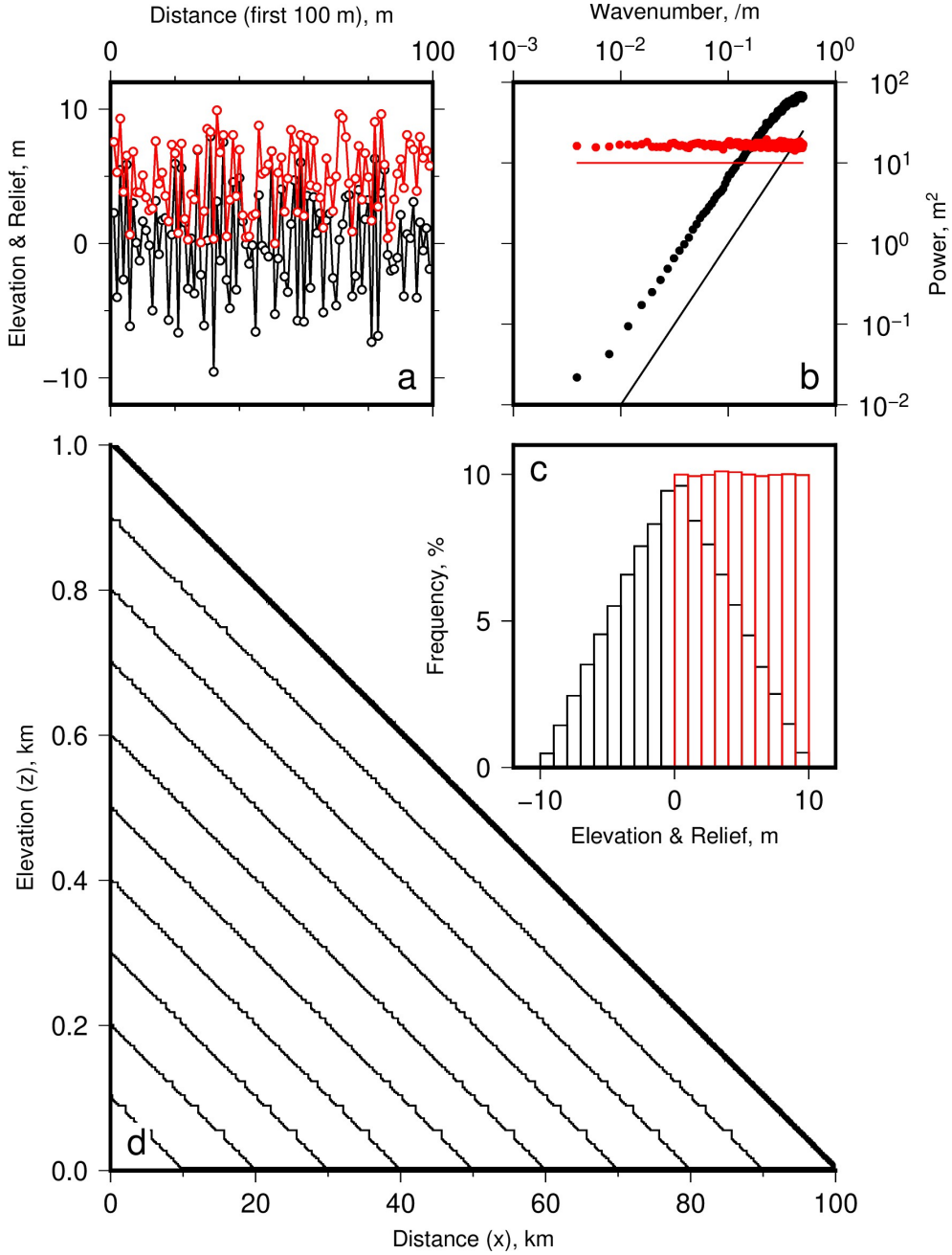| Parameter	Notation	Value	Unit
Density of water	ρ_w	1	$\times 10^3$ kg m^{-3}
Drag coefficient	C_d	O(1)	Dimensionless
Velocity of water	u	O(1–10)	m s^{-1}
Height of column facing water	h_1	O(1–10)	m
Gravitational acceleration	g	9.81	m s^{-2}
Depth of flowing water	h_2	O(1–10)	m
Average slope	$\mathrm{d}z/\mathrm{d}x$	O($10^{-3} - 10^{-2}$)	Dimensionless
Width of rock column	L	O(1)	m
Displaced water	h_3	O(1–10)	m
Density of rock	ρ_r	2–3	$\times 10^3$ kg m^{-3}
Height of rock column	H	O(1–10)	m
Elevation	z	O(1–1000)	m
Change in elevation between adjacent columns	Δz	O(1–10)	m

Figure 2: **Example of a 'large' model run.** (a) Red = Random uniformly distributed elevations, $z(x)$, added to the the linear slope shown in panel (d) to generate the starting condition, note that only first 100 m are shown for clarity. Black = local relief, i.e. $\Delta z = z_t^x - z_t^{x-1}$. (b) Power spectrum (from Fast Fourier Transform) of elevation (red circles) used to generate the random noise in the starting condition and relief (black circles). Note elevation spatial series has a white noise spectrum (solid red line), consistent with short wavelength ($\lesssim 100$ km) spectra of some real rivers (Roberts et al. 2019; Wapenhans et al., 2021). Black solid line = power $\propto k^2$, where k is wavenumber. (c) Histogram showing distributions of elevations (red) and relief in the starting condition (black). (d) 100-km-long river profile, containing 10^5 (1 m wide) blocks, evolving for 10^5 time steps. Thick black line = starting condition, thin lines = predicted profile every 10^4 time steps. Threshold, $c = 0.5$ m in this example. If block toppling occurs at a rate of 1 /year to 1 /century this model represents 10^5 to 10^7 years of evolution.
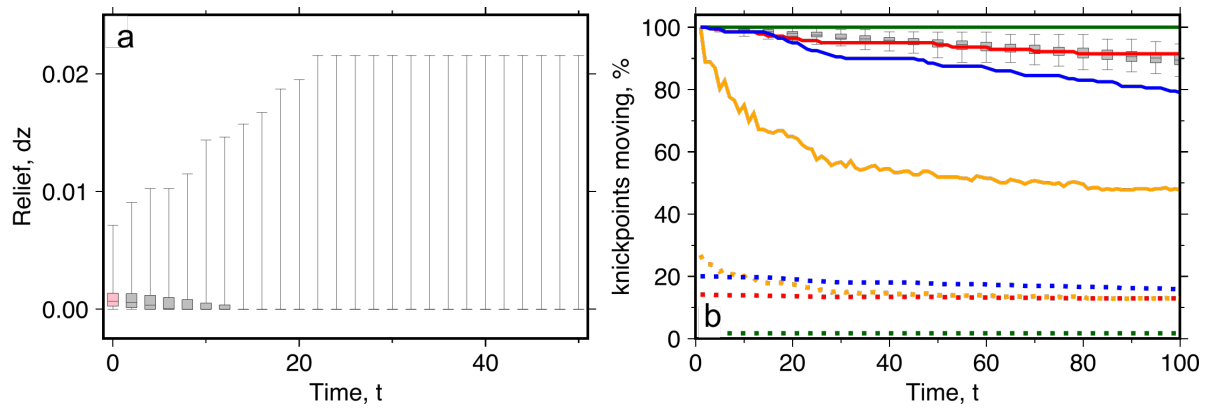
Figure 3: **Changing critical threshold, c, values.** (a) Distribution of relief as a function of time for simple linear model shown in Figure 1d–i of main manuscript; box and whiskers show extrema, median, 1st and 3rd quartile. Pink = distribution at first time step. (b) Solid curves shows percentage of knickpoints moving as a function of time relative to the number of knickpoints moving at first time step. Curves show results for different distributions of c; gray box and whiskers show distribution of values for constant value of c and 100 random distributions of starting condition (panel a and Figure 1d-i in main Ms); green/red = results for high/low constant value of c; blue = $c \propto 1/x$; orange = results for random uniform distribution of $c(x,t)$. Dotted curves = number of knickpoints moving as percentage of all relief measurements.

Wong, M., and Parker, G., 2006, Re-analysis and correction of bedload relation of Meyer-Peter and Muller using their own database: Journal of Hydraulic Engineering, v. 132, p. 1159–1168, https://doi.org/10.1061/(ASCE)0733-9429(2006)132:11(1159).